CHAPTER SIX

INTERPOLATING BETWEEN DATA POINTS

In this chapter we will study linear interpolation and Newton's forward and backward formulae which are used for readings at equal intervals.

. Interpolation is a procedure for determining a value of y corresponding to some specified value of x, when x falls between two known values, say xi and xi+1.

. The goal in interpolation is to determine the value of y as accurately as possible by passing an equation through the data points themselves rather than through the aggregate of the data points (as discussed in the last Chapter).

. The equation of interpolation need not be passed through all of the data points but just the data points within the immediate area of interest.

. The number of data points included in the equation generally depends on the nature of the interpolation curve (straight line or polynomial) and the level of accuracy required in the interpolated value.

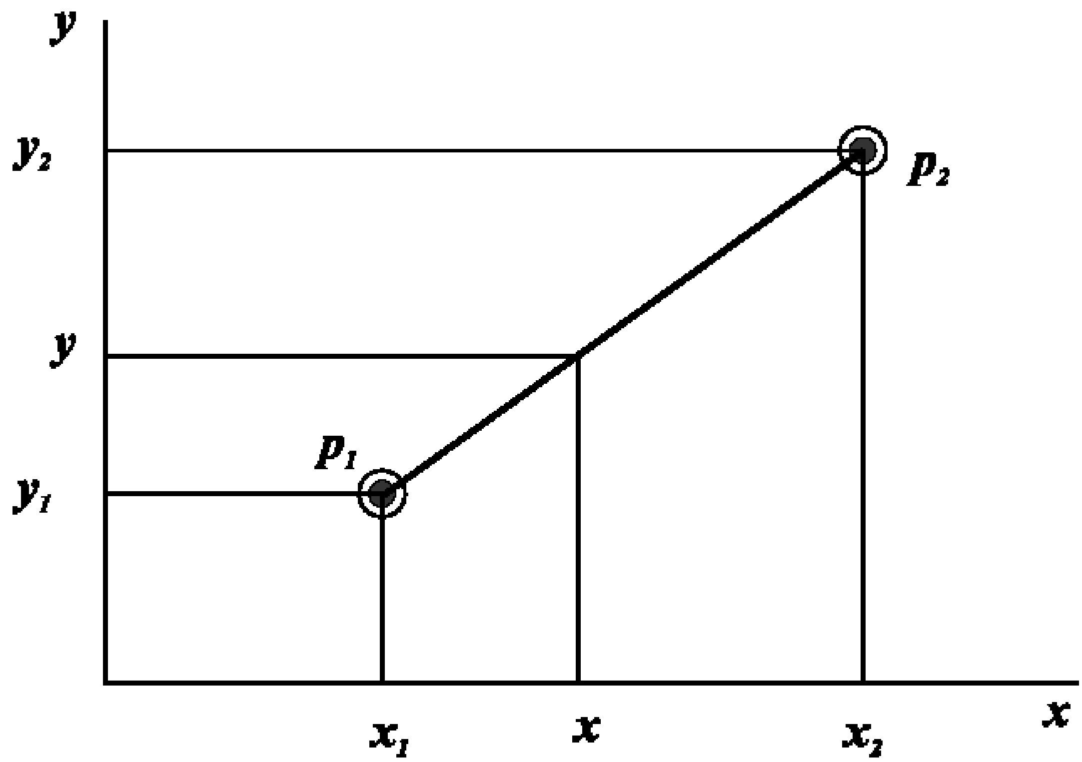
LINEAR INTERPOLATION

. The graph shown contains two data points:

$$p_1 = (x_1, y_1), \quad p_2 = (x_2, y_2)$$

and we wish to determine the value of *y* corresponding to some specified value of *x*, where

$$x1 < x < x2.$$



. By connecting the two data points p1 and p2 with a straight line, the desired value of y at the specified value of x can be determined as:

$$y = y_1 + \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

Example:

Given the two data points (2.0, 5.5) and (2.1, 7.2). Determine the value of *y* corresponding to *x* = 2.03 using linear interpolation.

Solution: We have,

x1 = 2.0   y1 = 5.5

x2 = 2.1   y2 = 7.2

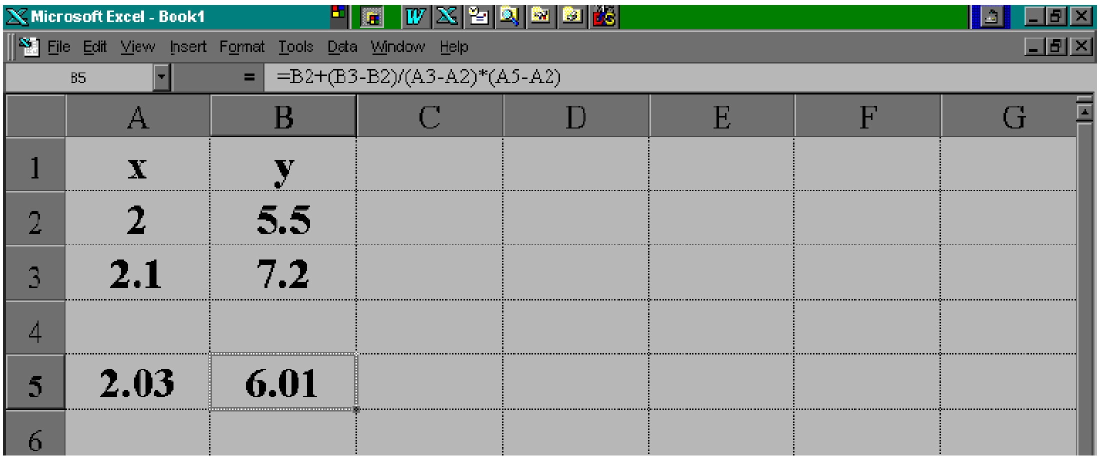
x = 2.03

Substituting in Eq. (6.1), we get

y(2.03) = 5.5 + [(7.2 - 5.5) / (2.1 – 2.0)] * (2.03 – 2.0) = 6.01

Thus, the new interpolated data point is (2.03, 6.01).

Solution of the Example using Excel:



POLYNOMIAL INTERPOLATION

a) FORWARD DIFFERENCES

. Suppose we wish to fit an interpolating polynomial through a number of equally spaced data points rather than a linear interpolation, as shown in the Figure below.

. Forward difference is defined as:

$$\Delta y_i = y_{i+1} - y_i$$

It is called the forward difference because it is obtained by subtracting the current point (yi) from the forward point (yi+1).

We can extend this concept to higher differences by writing

$$\Delta^2 y_i = \Delta y_{i+1} - \Delta y_i$$

$$\Delta^3 y_i = \Delta^2 y_{i+1} - \Delta^2 y_i$$

$$\cdots\cdots$$

$$\Delta^k y_i = \Delta^{k-1} y_{i+1} - \Delta^{k-1} y_i$$

Thus we can construct the following table of forward differences for a data set consisting of six data points:

x_i	y_i	Δy_i	$\Delta^2 y_i$	$\Delta^3 y_i$	$\Delta^4 y_i$	$\Delta^5 y_i$
x_1	y_1	Δy_1	$\Delta^2 y_1$	$\Delta^3 y_1$	$\Delta^4 y_1$	$\Delta^5 y_1$
x_2	y_2	Δy_2	$\Delta^2 y_2$	$\Delta^3 y_2$	$\Delta^4 y_2$	
x_3	y_3	Δy_3	$\Delta^2 y_3$	$\Delta^3 y_3$		
x_4	y_4	Δy_4	$\Delta^2 y_4$			
x_5	y_5	Δy_5				
x_6	y_6					

. The interpolated value y at a specified x can be given by using the *Gregory-Newton forward interpolating* formula given as:

$$y = y_1 + \frac{u}{h}\Delta y_1 + \frac{u(u-h)}{2!h^2}\Delta^2 y_1 + \frac{u(u-h)(u-2h)}{3!h^3}\Delta^3 y_1 + \ldots\ldots$$

where:

h **= equal spacing between the given** ***x*** **values (e.g.,** $h = x2 - x1$ **)**

$u = x - x1$

<u>Example:</u>

Use the forward interpolation Equation to interpolate the value of ***y*** **corresponding to** $x = 1.7$

for the given set of data:

x_i	y_i
1	1
2	2
3	10
4	44
5	141
6	366

***Solution*: We start the solution by constructing the forward difference table.**

x_i	y_i	Δy_i	$\Delta^2 y_i$	$\Delta^3 y_i$	$\Delta^4 y_i$	$\Delta^5 y_i$
1	1	1	7	19	18	10
2	2	8	26	37	28	
3	10	34	63	65		
4	44	97	128			
5	141	225				
6	366					

$h = 1$

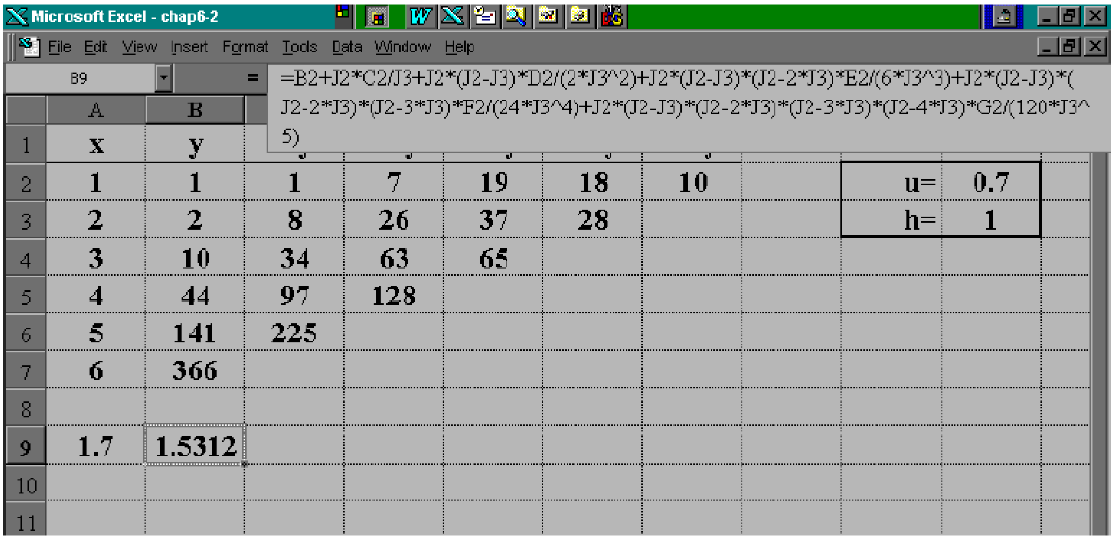
$x = 1.7$ $\quad x_1 = 1$ $\quad u = (x - x_1) = 0.7$

$y_1 = 1$ $\quad \Delta y_1 = 1$ $\quad \Delta^2 y_1 = 7$ $\quad \Delta^3 y_1 = 19$ $\quad \Delta^4 y_1 = 18$ $\quad \Delta^5 y_1 = 10$

Substituting these values into the Equation, we obtain:

$y\,(1.7) = 1.5312475$

Solution of the Example using Excel:



Microsoft Excel - chap6-2

B9 = =B2+J2*C2/J3+J2*(J2-J3)*D2/(2*J3^2)+J2*(J2-J3)*(J2-2*J3)*E2/(6*J3^3)+J2*(J2-J3)*(J2-2*J3)*(J2-3*J3)*F2/(24*J3^4)+J2*(J2-J3)*(J2-2*J3)*(J2-3*J3)*(J2-4*J3)*G2/(120*J3^5)

	A	B	C	D	E	F	G	H	I	J
1	x	y								
2	1	1	1	7	19	18	10		u=	0.7
3	2	2	8	26	37	28			h=	1
4	3	10	34	63	65					
5	4	44	97	128						
6	5	141	225							
7	6	366								
8										
9	1.7	1.5312								
10										
11										

b) BACKWARD DIFFERENCES

Forward difference interpolation works well when the interpolation point is near the beginning of the data set. On the other hand, backward differences provide a more accurate result when the interpolation point is near the end of the data set.

. **Backward difference is defined as:**

$$\nabla y_i = y_i - y_{i-1}$$

It is called the backward difference because it is obtained by subtracting the backward point (yi-1) from the current point (yi).

. **We can extend this concept to higher differences by writing**

$$\nabla^2 y_i = \nabla y_i - \nabla y_{i-1}$$

$$\nabla^3 y_i = \nabla^2 y_i - \nabla^2 y_{i-1}$$

……

$$\nabla^k y_i = \nabla^{k-1} y_i - \nabla^{k-1} y_{i-1}$$

Thus we can construct the following table of backward differences for a data set consisting of six data points:

x_i	y_i	∇y_i	$\nabla^2 y_i$	$\nabla^3 y_i$	$\nabla^4 y_i$	$\nabla^5 y_i$
x_1	y_1					
x_2	y_2	∇y_2				
x_3	y_3	∇y_3	$\nabla^2 y_3$			
x_4	y_4	∇y_4	$\nabla^2 y_4$	$\nabla^3 y_4$		
x_5	y_5	∇y_5	$\nabla^2 y_5$	$\nabla^3 y_5$	$\nabla^4 y_5$	
x_6	y_6	∇y_6	$\nabla^2 y_6$	$\nabla^3 y_6$	$\nabla^4 y_6$	$\nabla^5 y_6$

. **The interpolated value *y* at a specified *x* can be given by using the *Gregory-Newton backward interpolating* formula given as:**

$$\boxed{y = y_1 + \frac{u}{h}\nabla y_1 + \frac{u(u+h)}{2!h^2}\nabla^2 y_1 + \frac{u(u+h)(u+2h)}{3!h^3}\nabla^3 y_1 + \ldots\ldots}$$

Example:

Use Equation (6.3) to interpolate the value of *y* corresponding to *x* = 5.7 for the given set of data:

x_i	y_i
1	1
2	2
3	10
4	44
5	141
6	366

Solution**:** First we construct the backward difference table.

x_i	y_i	∇y_i	$\nabla^2 y_i$	$\nabla^3 y_i$	$\nabla^4 y_i$	$\nabla^5 y_i$
1	1					
2	2	1				
3	10	8	7			
4	44	34	26	19		
5	141	97	63	37	18	
6	366	225	128	65	28	10

$h = 1$

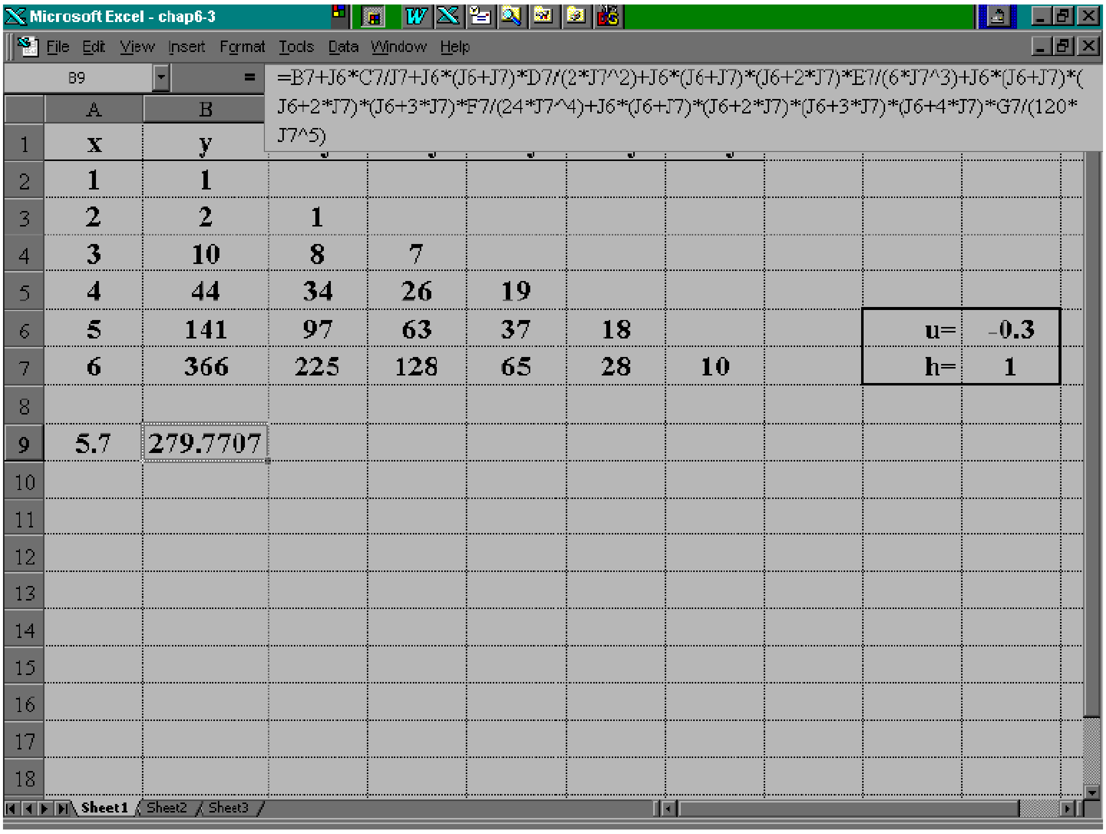
$x = 5.7$ $\quad x_1 = 6$ $\quad u = (x - x_1) = -0.3$

$y_1 = 366$ $\quad \nabla y_1 = 225$ $\quad \nabla^2 y_1 = 128$ $\quad \nabla^3 y_1 = 65$ $\quad \nabla^4 y_1 = 28$ $\quad \nabla^5 y_1 = 10$

Substituting these values into the backward interpolation Equation, we obtain:

y (5.7) = 279.7707

Solution of Example (3) using Excel:



Microsoft Excel - chap6-3

B9 =B7+J6*C7/J7+J6*(J6+J7)*D7/(2*J7^2)+J6*(J6+J7)*(J6+2*J7)*E7/(6*J7^3)+J6*(J6+J7)*(J6+2*J7)*(J6+3*J7)*F7/(24*J7^4)+J6*(J6+J7)*(J6+2*J7)*(J6+3*J7)*(J6+4*J7)*G7/(120*J7^5)

	A	B							
1	x	y							
2	1	1							
3	2	2	1						
4	3	10	8	7					
5	4	44	34	26	19				
6	5	141	97	63	37	18		u=	-0.3
7	6	366	225	128	65	28	10	h=	1
8									
9	5.7	279.7707							